Utility Functions
Case Study - Investor Risk Aversion Coefficient

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In this white paper we will estimate the risk aversion coefficient held by the average market investor, which is commonly thought to be between two and four. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We will use the following go-forward model assumptions...

Table 1: Model Assumptions

Description	Value	Notes
Share price at time zero ($)	25.00	
Expected return - mean (%)	9.50	S&P 500 historical return
Expected return - volatility (%)	18.00	S&P 500 historical volatility
Dividend yield (%)	2.50	S&P 500 dividend yield
Risk-free interest rate (%)	4.25	Approximate risk-free US treasury yield
Term in years (#)	3.00	

Questions:
1. Construct a table of probability-weighted portfolio values at time T.
2. What is the coefficient of risk aversion that is consistent with our share price at time zero?
3. Prove your answer to Question two above.

Portfolio Equations

We will define the variable S_t to be share price at time t, the variable μ to be expected annual return over the time interval $[0,T]$, the variable ϕ to be the annual dividend yield, the variable σ to be annual price volatility,and the variable Z to be a normally-distributed random number with mean zero and variance one. The equation for random share price at time T is... [3]

$$S_T = S_0 \,\mathrm{Exp}\left\{\left(\mu - \phi - \frac{1}{2}\,\sigma^2\right)T + \sigma\,\sqrt{T}\,Z\right\} \text{ ...where... } Z \sim N\Big[0,1\Big] \tag{1}$$

Using Equation (1) above, the equation for expected share price at time T is...

$$\mathbb{E}\Big[S_T\Big] = \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\} S_0\,\mathrm{Exp}\left\{\left(\mu - \phi - \frac{1}{2}\,\sigma^2\right)T + \sigma\,\sqrt{T}\,Z\right\}\delta Z = S_0\,\mathrm{Exp}\left\{\Big(\mu - \phi\Big)T\right\} \tag{2}$$

We will define the variable θ to be the realized annualized return over the time interval $[0,T]$. The equation for realized return is...

$$\theta = \ln\left(\frac{S_T}{S_0}\right)\Big/ T \tag{3}$$

We will define the variable C_t to be the cash account balance at time t and the variable Δ to be the risk-free interest rate. Dividends paid are deposited into the cash account and earn interest at the risk-free rate. Using Equation (3) above, the equation for random cash account balance at time T is... [4]

$$C_T = \phi\,(\theta - \Delta)^{-1} S_0 \operatorname{Exp}\left\{\Delta\,T\right\}\left(\operatorname{Exp}\left\{(\theta - \Delta)\,T\right\} - 1\right) \tag{4}$$

We will define the variable P_t to be portfolio value at time t. Our portfolio consists of one share of stock plus the dividend cash account. Using Equations (1) and (4) above, the equation for random portfolio value at time T is...

$$P_T = S_T + C_T \tag{5}$$

Exponential Utility Function

We will define the variable W_i to be scaled random wealth via the i'th probability-weighted scenario. Using Equation (5) above, the equation for scaled random wealth at time T is...

$$W_i = \frac{\text{End-of-term portfolio value at time } T}{\text{Initial investment at time zero}} = \frac{P_T}{S_0} \tag{6}$$

We will define the variable α to be a scalar whose value is greater than zero. Using Equation (6) above, the equations for the exponential untility function at its derivatives with respect to the scalar α are... [2]

$$U(W_i) = 1 - \operatorname{Exp}\left\{-\,\alpha\,W_i\right\} \text{ ...where... } U'(W_i) = \alpha \operatorname{Exp}\left\{-\,\alpha\,W_i\right\} \text{ ...and... } U''(W_i) = -\alpha^2 \operatorname{Exp}\left\{-\,\alpha\,W_i\right\} \tag{7}$$

We will define the variable λ to be the Arrow-Pratt measure of risk aversion. Using Equation (7) above and the data in Table 1 above, the equation for this measure of risk aversion is... [2]

$$\lambda = -\frac{U''(W_i)}{U'(W_i)} = -\frac{-\alpha^2 \operatorname{Exp}\{-\alpha\,W_i\}}{\alpha \operatorname{Exp}\{-\alpha\,W_i\}} = \alpha \text{ ...such that... } \alpha = \lambda = 3.00 \tag{8}$$

We will define the variable p_i to be the probability of realizing the i'th scenario. Using Equation (7) above, the equation for the expected utility of scaled wealth is... [2]

$$\mathbb{E}\left[U(\text{Scaled wealth})\right] = \sum_{i=1}^{n} p_i\,U(W_i) = \sum_{i=1}^{n} p_i\left(1 - \operatorname{Exp}\left\{-\,\alpha\,W_i\right\}\right) = 1 - \sum_{i=1}^{n} p_i \operatorname{Exp}\left\{-\,\alpha\,W_i\right\} \tag{9}$$

Note that scenario probabilities sum to one. This statement in equation form is...

$$\sum_{i=1}^{n} p_i = 1.00 \tag{10}$$

Solving For The Risk Aversion Coefficient

We will define the variable CE to be the value of the certainty equivalent at time T. The dollar value of the certainty equivalent is such that the utility of the certainty equivalent is equal to the utility of expected wealth. This statement in equation form is... [1]

$$U(CE) = \mathbb{E}\left[U(W)\right] \tag{11}$$

The equation for the certainty equivalent at time T is... [2]

$$CE = S_0 \operatorname{Exp}\left\{\Delta\,T\right\} \tag{12}$$

Using Equations (7) and (9) above, we can rewrite Equation (11) above as...

$$1 - \operatorname{Exp}\left\{-\,\alpha\,CE\right\} = \mathbb{E}\left[U(\text{Scaled wealth})\right] = 1 - \sum_{i=1}^{n} p_i \operatorname{Exp}\left\{-\,\alpha\,W_i\right\} \tag{13}$$

Note that we can rewrite Equation (13) above as...

$$\sum_{i=1}^{n} p_i \operatorname{Exp}\left\{-\,\alpha\,W_i\right\} - \operatorname{Exp}\left\{-\,\alpha\,CE\right\} = 0 \tag{14}$$

Our goal is to solve Equation (14) above for the scalar α.

The Answers To Our Hypothetical Problem

Using Table (1) above, the continuous-time model assumptions are...

Table 2: Continuous Time Model Assumptions

Symbol	Description	Value	Notes
S_0	Share price at time zero	25.0000	No adjustment needed
μ	Expected return - mean	0.0908	$\ln(1+0.0950)$
σ	Expected return - volatility	0.1800	No adjustment needed
ϕ	Dividend yield	0.0231	$\mu - \ln(1+0.0950-0.0250)$
Δ	Risk-free interest rate	0.0416	$\ln(1+0.0425)$
T	Term in years	3.0000	No adjustment needed

1. Construct a table of probability-weighted portfolio values at time T.

Bucket	Normal Random Numbers			EOT Portfolio Value		
Number	From	To	Prob	Stock	Cash	Total
1	-4.00	-3.60	0.000127	8.92	1.16	10.09
2	-3.60	-3.20	0.000528	10.11	1.22	11.33
3	-3.20	-2.80	0.001868	11.45	1.29	12.74
4	-2.80	-2.40	0.005642	12.97	1.36	14.33
5	-2.40	-2.00	0.014553	14.69	1.44	16.13
6	-2.00	-1.60	0.032049	16.64	1.52	18.17
7	-1.60	-1.20	0.060270	18.85	1.61	20.47
8	-1.20	-0.80	0.096786	21.36	1.71	23.07
9	-0.80	-0.40	0.132723	24.20	1.82	26.01
10	-0.40	0.00	0.155422	27.41	1.93	29.34
11	0.00	0.40	0.155422	31.05	2.06	33.11
12	0.40	0.80	0.132723	35.17	2.19	37.37
13	0.80	1.20	0.096786	39.85	2.34	42.18
14	1.20	1.60	0.060270	45.14	2.50	47.64
15	1.60	2.00	0.032049	51.13	2.68	53.81
16	2.00	2.40	0.014553	57.92	2.87	60.79
17	2.40	2.80	0.005642	65.62	3.08	68.69
18	2.80	3.20	0.001868	74.33	3.30	77.64
19	3.20	3.60	0.000528	84.21	3.55	87.76
20	3.60	4.00	0.000127	95.39	3.83	99.22

Example: Row 14

Using Equation (1) above and the model parameters in Table 2 above, random share price at time T is...

$$S_3 = 25.00 \times \mathrm{Exp}\left\{\left(0.0908 - 0.0231 - \frac{1}{2} \times 0.1800^2\right) \times 3.00 + 0.1800 \times \sqrt{3.00} \times \frac{1.20+1.60}{2}\right\} = 45.14 \tag{15}$$

Using Equations (3) and (15) above, the random return over the time interval $[0,3]$ is...

$$\theta = \ln\left(\frac{45.14}{25.00}\right) \Big/ 3.00 = 0.1970 \tag{16}$$

Using Equation (4) and (16) above and the model parameters in Table 2 above, random cash balance at time T is...

$$C_3 = 0.0231 \times (0.1970 - 0.0416)^{-1} \times 25.00 \times \mathrm{Exp}\left\{0.0416 \times 3.00\right\} \times \left(\mathrm{Exp}\left\{(0.1970 - 0.0416) \times 3.00\right\} - 1\right) = 2.50 \tag{17}$$

The equation for the probability of realizing Equations (15) and (17) above is...

$$\text{Probability} = NORMSDIST(1.60) - NORMSDIST(1.20) = 0.060270 \tag{18}$$

2. What is the coefficient of risk aversion that is consistent with our share price at time zero?

Using Equations (12) above and the model parameters in Table 2 above, the equation for the certainty equivalent is...

$$CE = 25.00 \times \text{Exp}\bigg\{0.0416 \times 3.00\bigg\} = 28.32 \tag{19}$$

Using Equation (14) above, the itereated solution to the value of the scalar alpha is... [5]

	Alpha Range				F(Alpha)		New
Iteration	A	B	C	F(A)	F(B)	F(C)	Range
1	2.00000	3.00000	4.00000	0.00810	-0.00096	-0.00253	ab
2	2.00000	2.50000	3.00000	0.00810	0.00211	-0.00096	bc
3	2.50000	2.75000	3.00000	0.00211	0.00029	-0.00096	bc
4	2.75000	2.87500	3.00000	0.00029	-0.00040	-0.00096	ab
5	2.75000	2.81250	2.87500	0.00029	-0.00007	-0.00040	ab
6	2.75000	2.78125	2.81250	0.00029	0.00010	-0.00007	bc
7	2.78125	2.79688	2.81250	0.00010	0.00001	-0.00007	bc
8	2.79688	2.80469	2.81250	0.00001	-0.00003	-0.00007	ab
9	2.79688	2.80078	2.80469	0.00001	-0.00001	-0.00003	ab
10	2.79688	2.79883	2.80078	0.00001	0.00000	-0.00001	bc
11	2.79883	2.79980	2.80078	0.00000	0.00000	-0.00001	ab
12	2.79883	2.79932	2.79980	0.00000	0.00000	0.00000	bc
13	2.79932	2.79956	2.79980	0.00000	0.00000	0.00000	ab
14	2.79932	2.79944	2.79956	0.00000	0.00000	0.00000	bc
15	2.79944	2.79950	2.79956	0.00000	0.00000	0.00000	ab

Notes:
1. At iteration [1] we set A and C to be the lower and upper bound, respectively, of the expected risk aversion coefficient range $[2, 4]$.
2. The value of B is A plus C divided by 2.
3. The values of F(A), F(B) and F(C) is Equation (14) above with A, B and C as the values of alpha.
4. The column New Range is the new range of the scalar alpha based on the Bisection method of solving non-linear equations. [5]

The iterated value of the scalar alpha that is consistent with a share price at time zero of \$25.00 is...

$$\text{Value of alpha after 15'th iteration} = 2.79950 \tag{20}$$

3. Prove your answer to Question two above.

Using Equations (7) and (20) above, the utility of the certainty equivalent at time T is...

$$U(CE) = 1 - \text{Exp}\bigg\{-2.79950 \times \frac{28.32}{25.00}\bigg\} = 0.958073 \tag{21}$$

Using Equations (9) and (20) above and the portfolio value Table from question one above, the utility of scaled wealth at time T is...

$$\mathbb{E}\bigg[U(\text{Scaled wealth})\bigg] = 1 - \sum_{i=1}^{n} p_i \,\text{Exp}\bigg\{-2.79950 \times \frac{W_i}{25.00}\bigg\} = 0.958073 \tag{22}$$

Given that the results of Equations (21) and (22) above are the same, the risk aversion coefficient obtained via Equation (20) is correct.

References

[1] Gary Schurman, *Introduction To Utility Funtions,* October, 2023.

[2] Gary Schurman, *The Exponential Utility Funtion,* October, 2023.

[3] Gary Schurman, *Brownian Motion - Introduction to Stochastic Calculus,* February, 2012.

[4] Gary Schurman, *Retaining And Reinvesting Dividends,* December, 2020.

[5] Gary Schurman, *Solving For IRR Via The Bisection Method,* August, 2023.